Chapter Eight

Circle

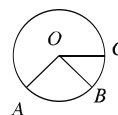
We have already known that a circle is a geometrical figure in a plane consisting of points equidistant from a fixed point. Different concepts related to circles like centre, diameter, radius, chord etc has been discussed in previous class. In this chapter, the propositions related to arcs and tangents of a circle in the plane will be discussed.

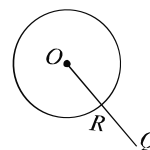
At the end of the chapter, the students will be able to

- Explain arcs, angle at the centre, angle in the circle, quadrilaterals inscribed in the cirlce
- Prove theorems related to circle
- State constructions related to circle.

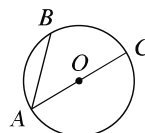
8·1 Circle

A circle is a geometrical figure in a plane whose points are equidistant from a fixed point. The fixed point is the centre of the circle. The closed path traced by a point that keeps it distance from the fixed centre is a circle. The distance from the centre is the radius of the circle.

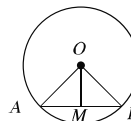
Let O be a fixed point in a plane and r be a fixed measurement. The set of points which are at a distance r from O is the circle with centre O and radius r. In the figure, O is the centre of the circle and A, B and C are three points on the circle. Each of OA, OB and OC is a radius of the circle. Some coplanar points are called concylcic if a circle passes through these points, i.e. there is a circle on which all these points lie. In the above figure, the points A, B and C are concyclic.

Interior and Exterior of a Circle

If O is the centre of a circle and r is its radius, the set of all points on the plane whose distances from O are less than r, is called the interior region of the circle and the set of all points on the plane whose distances from O are greater than r, is called the exterior region of the circle. The line segment joining two points of a circle lies inside the circle.

The line segment drawn from an interior point to an exterior point of a circle intersects a circle at one and only one point. In the figure, P and Q are interior and exterior points of the circle respectively. The line segment PQ intersects the circle at R only.

Chord and Diameter of a Circle

The line segment connecting two different points of a circle is a chord of the circle. If the chord passes through the centre it is known as diameter. That is, any chord forwarding to the centre of the circle is diameter. In the figure, AB and AC are two chords and O is the centre of the circle. The chord AC is a diameter, since it passes through the centre. OA and OC are two radii of the circle. Therefore, the centre of a circle is the mid-point of any diameter. The length of a diameter is $2r$, where r is the radius of the circle.

Theorem 1

The line segment drawn from the centre of a circle to bisect a chord other than diameter is perpendicular to the chord.

Let AB be a chord (other than diameter) of a circle ABC with centre O and M be the midpoint of the chord. Join O, M. It is to be proved that the line segment OM is perpendicular to the chord AB.

Construction: Join O, A and O, B.

Proof:

Steps	Justification
(1) In $\Delta\angle OAM$ and ΔOBM,	
$OA = OB$	[M is the mid point of AB]
$AM = BM$	[radius of same circle]
and $OM = OM$	[common side]
Therefore, $\Delta OAM \cong \Delta OBM$	[SSS theorem]
$\therefore \angle OMA = \angle OMB$	
(2) Since the two angles are equal and together maka a straight angle.	
$\angle OMA = \angle OMB = 1$ right angle.	
Therefore, $OM \perp AB$. (Proved).	

Corollary 1: The perpendicular bisector of any chord passes through the centre of the circle.

Corollary 2: A straight line can not intersect a circle in more than two points.

Activity :

1. The theorem opposite of the theorem 1 states that the perpendicular from the centre of a circle to a chord bisects the chord. Prove the theorem.

Theorem 2

All equal chords of a circle are equidistant from the centre.

Let AB and CD be two equal chords of a circle with centre O. It is to be proved that the chords AB and CD are equidistant from the centre.

Construction: Draw from O the perpendiculars OE and OF to the chords AB and CD respectively. Join O, A and O, C.

Proof:

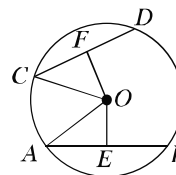
Steps	Justification
(1) $OE \perp AB$ and $OF \perp CD$ Therefore, $AE = BE$ and $CF = BF$. $\therefore AE = \frac{1}{2} AB$ and $CF = \frac{1}{2} CD$	[The perpendicular from the centre bisects the chord]
(2) But $AB = DC$ $\therefore AE = CF$	[supposition]
(3) Now in the right-angled triangles ΔOAE and ΔOCF hypotenuse OA = hypotenuse OC and $AE = CF$ $\therefore \Delta OAE \cong \Delta OCF$ $\therefore OE = OF$	[radius of same circle] [Step 2] [RHS theorem]
(4) But OE and OF are the distances from O to the chords AB and CD respectively. Therefore, the chords AB and CD are equidistant from the centre of the circle. (Proved)	

Theorem 3

Chords equidistant from the centre of a circle are equal.

Let AB and CD be two chords of a circle with centre O. OE and OF are the perpendiculars from O to the chords AB and CD respectively. Then OE and OF represent the distance from centre to the chords AB and CD respectively.

It is to be proved that if $OE = OF$, $AB = CD$.

Construction: Join O, A and O, C.

Proof:

Steps	**Justification**
(1) Since $OE \perp AB$ and $OF \perp CD$. Therefore, $\angle OEA = \angle OFC =$ 1 right angle	[right angles]
(2) Now in the right angled triangles ΔOAE and ΔOCF	
hypotenuse OA = hypotenuse OC and $OE = OF$	[radius of same circle]
$\therefore \quad \Delta OAE \cong \Delta OCF$ $\therefore \quad AE = CF$.	[RS theorem]
(3) $AE = \dfrac{1}{2} AB$ and $CF = \dfrac{1}{2} CD$	[The perpendicular from the centre bisects the chord]
(4) Therefore $\dfrac{1}{2} AB = \dfrac{1}{2} CD$ i.e., $AB = CD$ (Proved)	

Corollary 1: The diameter is the greatest chord of a circle.

Exercise 8.1

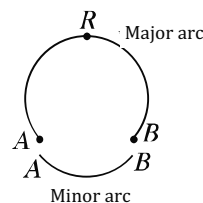
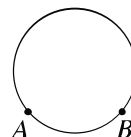
1. Prove that if two chords of a circle bisect each other, their point of intersection is the centre of the circle.
2. Prove that the straight line joining the middle points of two parallel chords of a circle pass through the centre and is perpendicular to the chords.
3. Two chords AB and AC of a circle subtend equal angles with the radius passing through A. Prove that, $AB = AC$.
4. In the figure, O is the centre of the circle and chord AB = chord AC. Prove that $\angle BAO = \angle CAO$.
5. A circle passes through the vertices of a right angled triangle. Show that, the centre of the circle is the middle point of the hypotenuse.
6. A chord AB of one of the two concentric circles intersects the other circle at points C and D. Prove that, $AC = BD$.
7. If two equal chords of a circle intersect each other, show that two segments of one are equal to two segments of the other.
8. Prove that, the middle points of equal chords of a circle are concyclic.
9. Show that, the two equal chords drawn from two ends of the diameter on its opposite sides are parallel.

10. Show that, the two parallel chords of a circle drawn from two ends of a diameter on its opposite sides are equal.

11. Show that, of the two chords of a circle the bigger chord is nearer to the centre than the smaller.

8.2 The arc of a circle

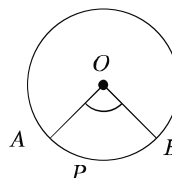
An arc is the piece of the circle between any two points of the circle. Look at the pieces of the circle between two points *A* and *B* in the figure. We find that there are two pieces, one comparatively large and the other small. The large one is called the *major arc* and the small one is called the *minor arc.*

A and *B* are the terminal points of this arc and all other points are its internal points. With an internal fixed point *C* the arc is called arc *ABC* and is expressed by the symbol *ACB*. Again, sometimes minor arc is expressed by the symbol *AB*. The two points *A* and *B* of the circle divide the circle into two arcs. The terminal points of both arcs are *A* and *B* and there is no other common point of the two arcs other than the terminal points.

Arc cut by an Angle

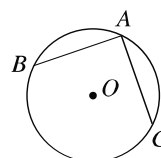
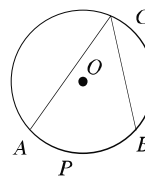
An angle is said to cut an arc of a circle if

(i) each terminal point of the arc lies on the sides of the angle

(ii) each side of the angle contains at least one terminal point

(iii) Every interior point of the arc lies inside the angle. The angle shown in the figure cuts the *APB* arc of the circle with centre O.

Angle in a Circle

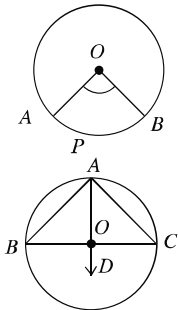
If the vertex of an angle is a point of a circle and each side of the angle contains a point of the circle, the angle is said to be an angle in the circle or an angle inscribed in the circle. The angles in the figure are all angles in a circle. Every angle in a circle cuts an arc of the circle. This arc may be a major or minor arc or a semicircle.

The angle in a circle cuts an arc of the circle and the angle is said to be standing on the cut off arc. The angle is also known as the angle inscribed in the conjugate arc. In the adjacent figure, the angle stands on the arc *APB* and is inscribed in the conjugate arc *ACB*. It is to be noted that *APB* and *ACB* are mutually conjugate.

Remark: The angle inscribed in an arc of a circle is the angle with vertex in the arc and the sides passing through the terminal points of the arc. An angle standing on an arc is the angle inscribed in the conjugate arc.

Angle at the Centre

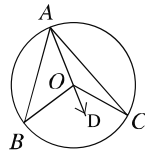
The angle with vertex at the centre of the circle is called an angle at the centre. An angle at the centre cuts an arc of the circle and is said to stand on the arc. In the adjacent figure, $\angle AOB$ is an angle at the centre and it stands on the arc *APB*.

Every angle at the centre stands on a minor arc of the circle. In the figure *APB* is the minor arc. So the vertex of an angle at the centre always lies at the centre and the sides pass through the two terminal points of the arc.

To consider an angle at the centre standing on a semi-circle the above description is not meaningful. In the case of semi-circle, the angle at the centre $\angle BOC$ is a straight angle and the angle on the arc $\angle BAC$ is a right angle.

Theorem 4

The angle subtended by the same arc at the centre is double of the angle subtended by it at any point on the remaining part of the circle.

Given an arc *BC* of a circle subtending angles $\angle BOC$ at the centre *O* and $\angle BAC$ at a point *A* of the circle *ABC*. We need to prove that $\angle BOC = 2\ \angle BAC$.

Construction: Suppose, the line segment *AC* does not pass through the centre *O*. In this case, draw a line segment *AD* at *A passing through the centre O*.

Proof :

Steps	Justification
(1) In ΔAOB, the external angle $\angle BOD = \angle BAO + \angle ABO$	[An exterior angle of a triangle is equal to the sum of the two interior opposite angles.]
(2) Also in $\Delta\ AOB$, $OA = OB$	[Radius of a circle]
Therefore, $\angle BAO = \angle ABO$	[Base angles of an isosceles triangle are equal]
(3) From steps (1) and (2), $\angle BOD = 2\angle BAO$.	
(4) Similarly, $\angle\ COD = 2\ \angle\ CAO$	
(5) From steps (3) and (4), $\angle BOD + \angle COD = 2\angle BAO + 2\angle CAO$	[by adding]
This is the same as $\angle BOC = 2\angle BAC$. [Proved]	

We can state the theorem in a different way. The angle standing on an arc of the circle is half the angle subtended by the arc at the centre.

Activity : Prove the theorem 4 when AC passes through the centre of the circle ABC.

Theorem 5

Angles in a circle standing on the same arc are equal.

Let O be the centre of a circle and standing on the arc BD, $\angle BAD$ and $\angle BED$ be the two angles in the circle. We need to prove that $\angle BAD = \angle BED$.

Construction : Join O, B and O, D.

Proof :

Steps	Justification
(1) The arc BD subtends an angle $\angle BOD$ at the centre O. Therefore, $\angle BOD = 2\angle BAD$ and $\angle BOD = 2\angle BED$ $\therefore\ 2\angle BAD = 2\angle BED$ or, $\angle BAD = \angle BED$ (Proved)	[The angle subtended by an arc at the centre is double of the angle subtended on the circle]

Theorem 6

The angle in the semi- circle is a right angle.

Let AB be a diameter of circle with centre at O and $\angle ACB$ is the angle subtended by a semi-circle. It is to be proved that $\angle ACB$ = 1 right angle.

Construction: Take a point D on the circle on the opposite side of the circle where C is located.

Proof:

Steps	Justification
(1) The angle standing on the arc ADB $\angle ACB = \frac{1}{2}$ (straight angle in the centre $\angle AOB$) (2) But the straight angle $\angle AOB$ is equal to 2 right angles. $\angle ACB = \frac{1}{2}$ (2 right angles) = 1 right angle. (Proved)	[The angle standing on an arc at any point of the circle is half the angle at the centre]

Corollary 1. The circle drawn with hypotenuse of a right-angled triangle as diameter passes through the vertices of the triangle.

Corollary 2. The angle inscribed in the major arc of a circle is an acute angle.

Activity :
1 Prove that any angle inscribed in a minor arc is obtuse.

Exercise 8.2

1 *ABCD* is a quadrilateral inscribed in a circle with centre *O*. If the diagonals *AB* and *CD* intersect at the point *E*, prove that $\angle AOB + \angle COD = 2\angle AEB$.

2 Two chords *AB* and *CD* of the circle *ABCD* intersect at the point *E*. Show that, ΔAED and ΔBEC are equiangular.

3. In the circle with centre *O* $\angle ADB + \angle BDC = 1$ right angle. Prove that, *A, B* and *C* lie in the same straight line.

4. Two chords *AB* and *CD* of a circle intersect at an interior point. Prove that, the sum of the angles subtended by the arcs *AC* and *BD* at the centre is twice $\angle AEC$.

5. Show that, the oblique sides of a cyclic trapezium are equal.

6. *AB* and *CD* are the two chords of a circle ; *P* and *Q* are the middle points of the two minor arcs made by them. The chord *PQ* intersects the chords *AB* and *AC* at points *D* and *E* respectively. Show that, $AD = AE$.

8·3 Quadrilateral inscribed in a circle

An inscribed quadrilateral or a quadrilateral inscribed in a circle is a quadrilateral having all four vertices on the circle. Such quadrilaterals possess a special property. The following avtivity helps us understand this property.

Activity:
Draw a few inscribed quadrilaterals ABCD. This can easily be accomplished by drawing circles with different radius and then by taking four arbitrary points on each of the circles. Measure the angles of the quadrilaterals and fill in the following table.

Serial No.	$\angle A$	$\angle B$	$\angle C$	$\angle D$	$\angle A+\angle C$	$\angle B+\angle D$
1						
2						
3						
4						
5						

What to you infer from the table?

Circle related Theorems

Theorem 7

The sum of the two opposite angles of a quadrilateral inscribed in a circle is two right angles.

Let *ABCD* be a quadrilateral inscribed in a circle with centre *O*. It is required to prove that, $\angle ABC + \angle ADC = 2$ right angles and $\angle BAD + \angle BCD = 2$ right angles.

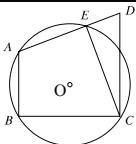
Construction : Join *O, A* and *O, C*.

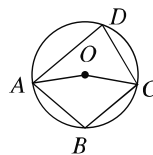
Proof :

Steps	Justification
(1) Standing on the same arc ADC, the angle at centre $\angle AOC = 2$ ($\angle ABC$ at the circumference) that is, $\angle AOC = 2 \angle ABC$.	[The angle subtended by an arc at the centre is double of the angle subtended by it at the circle]
(2) Again, standing on the same arc *ABC*, reflex $\angle AOC$ at the centre = 2 ($\angle ADC$ at the circumference) that is, reflex $\angle AOC = 2 \angle ADC$ $\therefore$ $\angle AOC$ + reflex $\angle AOC = 2(\angle ABC + \angle ADC)$ But $\angle AOC$ + reflex $\angle AOC = 4$ right angles $\therefore$ $2(\angle ABC + \angle ADC) = 4$ right angles $\therefore$ $\angle ABC + \angle ADC = 2$ right angles. In the same way, it can be proved that $\angle BAD + \angle BCD = 2$ right angles. (Proved)	[The angle subtended by an arc at the centre is double of the angle subtended by it at the circle]

Corollary 1: If one side of a cyclic quadrilateral is extended, the exterior angle formed is equal to the opposite interior angle.

Corollary 2: A parallelogram inscribed in a circle is a rectangle.

Theorem 8

If two opposite angles of a quadrilateral are supplementary, the four vertices of the quadrilateral are concyclic.

Let *ABCD* be the quadrilateral with $\angle ABC + \angle ADC = 2$ right angles, inscribed in a circle with centre O. It is required to prove that the four points *A, B, C, D* are concyclic.

Construction: Since the points *A,B, C* are not collinear, there exists a unique circle which passes through these three points. Let the circle intersect *AD* at *E*. Join *A,E*.

Proof :

Steps	Justification
(1) ABCE is a quadrilateral inscribed in the circle. Therefore, $\angle ABC + \angle AEC = 2$ right angles. But $\angle ABC + \angle ADC = 2$ right angles [given] $\therefore$ $\angle AEC = \angle ADC$ But this is impossible, since in ΔCED, exterior $\angle AEC >$ opposite interior $\angle ADC$ Therefore, *E* and *D* points can not be different points. So, *E* must coincide with the point *D*. Therefore, the points *A, B, C, D* are concyclic.	[The sum of the two opposite angles of an inscribed quadrilateral is two right angles.] [The exterior angle is greater than any opposite interior angle.]

Exercise 8.3

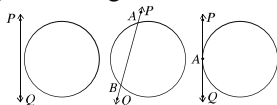
1. If the internal and extern al bisectors of the angles $\angle B$ and $\angle C$ of ΔABC meet at P and Q respectively, prove that *B, P, C, Q* are concyclcic.
2. Prove that, the bisector of any angle of a cyclic quadrilateral and the exterior bisector of its opposite angle meet on the circumference of the circle.
3. *ABCD* is a circle. If the bisectors of $\angle CAB$ and $\angle CBA$ meet at the point *P* and the bisectors of $\angle DBA$ and $\angle DAB$ meet at *Q*, prove that, the four points *A, Q, P, B* are concyclic.
4. The chords *AB* and *CD* of a circle with centre *D* meet at right angles at some point within the circle, prove that, $\angle AOD + \angle BOC = 2$ right angles.
5. If the vertical angles of two triangles standing on equal bases are supplementary, prove that their circumeircles are equal.
6. The opposite angles of the quadrilateral *ABCD* are supplementary to each other. If the line *AC* is the bisector of $\angle BAD$, prove that, $BC = CD$.

8·4 Secant and Tangent of the circle

Consider the relative position of a circle and a straight line in the plane. Three possible situations of the following given figures may arise in such a case:

(a) The circle and the straight line have no common points

(b) The straight line has cut the circle at two points

(c) The straight line has touched the circle at a point.

A circle and a straight line in a plane may at best have two points of intersection. If a circle and a straight line in a plane have two points of intersection, the straight line is called a secant to the circle and if the point of intersection is one and only one, the straight line is called a tangent to the circle. In the latter case, the common point is called the point of contact of the tangent. In the above figure, the relative position of a circle and a straight line is shown. In figure (i) the circle and the straight line PQ have no common point; in figure (ii) the line PQ is a secant, since it intersects the circle at two points A and B and in figure (iii) the line PQ has touched the circle at A. PQ is a tangent to the circle and A is the point of contact of the tangent.

Remark : All the points between two points of intersection of every secants of the circle lie interior of the circle.

Common tangent

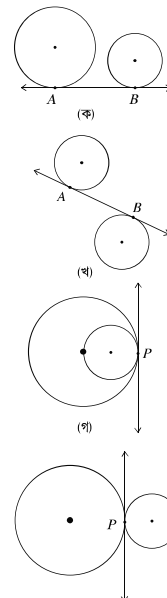
If a straight line is a tangent to two circles, it is called a common tangent to the two circles. In the adjoining figures, AB is a common tangent to both the circles. In figure (a) and (b), the points of contact are different. In figure (c) and (d), the points of contact are the same.

If the two points of contact of the common tangent to two circles are different, the tangent is said to be

(a) direct common tangent if the two centres of the circles lie on the same side of the tangent and

(b) transverse common tangent, if the two centres lie on opposite sides of the tangent.

The tangent in figure (a) is a direct common one and in figure (b) it is a transverse common tangent.

If a common tangent to a circle touches both the circles at the same point, the two circles are said to touch each other at that point. In such a case, the two circles are said to have touched internally or externally according to their centres lie on the same side or opposite side of the tangent. In figure (c) the two circles have touched each other internally and in figure (d) externally.

Theorem 9

The tangent drawn at any point of a circle is perpendicular to the radius through the point of contact of the tangent.

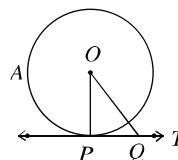
Let PT be a tangent at the point P to the circle with centre O and OP is the radius throug0h the point of contact. It is required to prove that, $PT \perp OP$.

Let PT be a tangent at the point P to the circle with centre O and OP is the radius throug0h the point of contact. It is required to prove that, $PT \perp OP$.

Construction: Take any point Q on PT and join O, Q.

Proof :

Since PT is a tangent to the circle at the point P, hence every point on it except P lies outside the circle. Therefore, the point Q is outside of the circle.

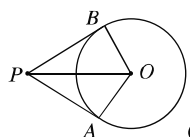
OQ is greater than OP that is, $OQ > OP$ and it is true for every point Q on the tangent PT except P. So, OP is the shortest distance from the centre O to PT. Therefore, $PT \perp OP$. (Proved)

Corollary 1. At any point on a circle, only one tangent can be drawn.

Corollary 2. The perpendicular to a tangent at its point of contact passes through the centre of the circle.

Corollary 3. At any point of the circle the perpendicular to the radius is a tangent to the circle.

Theorem 10

If two tangents are drawn to a circle from an external point, the distances from that point to the points of contact are equal.

Let P be a point outside a circle ABC with centre O, and two line segments PA and PB be two tangents to the circle at points A and B. It is required to prove that. $PA = PB$.

Construction: Let us join O, A; O, B and O, P.

Proof:

Steps	Justification
(1) Since PA is a tangent and OA is the radius through the point of tangent $PA \perp OA$. $\therefore \angle PAO =$ 1 right angle Similarly, $\angle PBO =$ 1 right angle $\therefore$ both ΔPAO and ΔPBO are right-angled triangles. (2) Now in the right angled triangles ΔPAO and ΔPBO, hypotenuse $PO =$ hypotenuse PO,	[The tangent is perpendicular to the radius through the point of contact of the tangent]

Remarks:

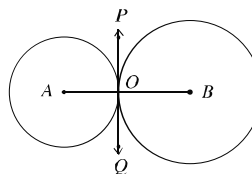
1 If two circles touch each other externally, all the points of one excepting the point of contact will lie outside the other circle.

2 If two circles touch each other internally, all the points of the smaller circle excepting the point of contact lie inside the greater circle.

Theorem 11

If two circles touch each other externally, the point of contact of the tangent and the centres are collinear.

Let the two circles with centres at A and B touch each other externally at O. It is required to prove that the points A, O and B are collinear.

Construction: Since the given circles touch each other at O, they have a common tangent at the point O. Now draw the common tangent POQ at O and join O, A and O, B.

Proof: In the circles OA is the radius through the point of contact of the tangent and POQ is the tangent.

Therefore $\angle POA =1$ right angle. Similarly $\angle POB =1$ right angle

Hence $\angle POA + \angle POB =1$ right angle $+1$ right angle $=2$ right angles

or $\angle AOB =2$ right angles i.e. $\angle AOB$ is a straight angle. $\therefore A, O$ and B are collinear. (Proved)

Corollary 1. If two circles touch each other externally, the distance between their centres is equal to the sum of their radii

Corollary 2. If two circles touch each other internally, the distance between their centres is equal to the difference of their radii.

Activity:

1 Prove that, if two circles touch each other internally, the point of contact of the tangent and the centres are collinear.

Exercise 8·4

1 Two tangents are drawn from an external point P to the circle with centre O. Prove that OP is the perpendicular bisector of the chord through the touch points.

2 Given that tangents PA and PB touches the circle with centre O at A and B respectively. Prove that PO bisects $\angle APB$.

3. Prove that, if two circles are concentric and if a chord of the greater circle touches the smaller, the chord is bisected at the point of contact.
4. *AB* is a diameter of a circle and *BC* is a chord equal to its radius. If the tangents drawn at *A* and *C* meet each other at the point *D*, prove that *ACD* is an equilateral triangle.
5. Prove that a circumscribed quadrilateral of a circle having the angles subtended by opposite sides at the centre are supplementary.

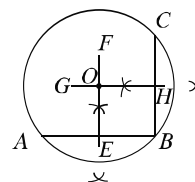
8·5 Constructions related to circles

Construction 1

To determine the centre of a circle or an arc of a circle.

Given a circle as in figure (a) or an arc of a circle as in figure (b). It is required to determine the centre of the circle or the arc.

Construction : In the given circle or the arc of the circle, three different points *A, B, C* are taken. The perpendicular bisectors *EF* and *GH* of the chords *AB* and *BC* are drawn respectively. Let the bisectors intersect at *O*. The *O* is the required centre of the circle or of the arc of the circle.

Proof: By construction, the line segments *EF* and *GH* are the perpendicular bisectors of chords *AB* and *BC* respectively. But both *EF* and *GH* pass through the centre and their common point is *O*. Therefore, the point *O* is the centre of the circle or of the arc of the circle.

Tangents to a Circle

We have known that a tangent can not be drawn to a circle from a point internal to it. If the point is on the circle, a single tangent can be drawn at that point. The tangent is perpendicular to the radius drawn from the specified point. Therefore, in order to construct a tangent to a circle at a point on it, it is required to construct the radius from the point and then construct a perpendicular to it. Again, if the point is located outside the circle, two tangents to the circle can be constructed.

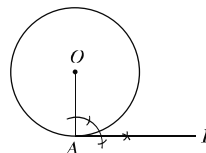
Construction 2

To draw a tangent at any point of a circle.

Let *A* be a point of a circle whose centre is *O*. It is required to draw a tangent to the circle at the point *A*.

Construction :

O, A are joined. At the point *A,* a perpendicular *AP* is drawn to *OA*. Then *AP* is the required tangent.

Proof: The line segment *OA* is the radius passing through *A* and *AP* is perpendicular to it. Hence, *AP* is the required tangent.

Remark : At any point of a circle only one tangent can be drawn.

Construction 3

To draw a tangent to a circle from a point outside.

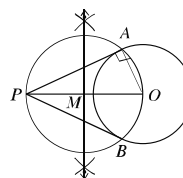
Let *P* be a point outside of a circle whose centre is *O*. A tangent is to be drawn to the circle from the point *P*.

Construction :

(1) Join *P*, *O*. The middle point *M* of the line segment *PO* is determined.

(2) Now with *M* as centre and *MO* as radius, a circle is drawn. Let the new circle intersect the given circle at the points *A* and *B*.

(3) *A*, *P* and *B*, *P* are joined.

Then both *AP* or *BP* are the required tangents.

Proof: *A, O* and *B, O* are joined. *PO* is the diameter of the circle *APB*.

$\therefore \angle PAO =$ 1 right angle [the angle in the semicircle is a right angle]

So the line segment *OA* is perpendicular to *AP*. Therefore, the line segment *AP* is a tangent at *A* to the circle with centre at *O*. Similarly the line segment *BP* is also a tangent to the circle.

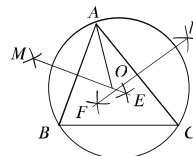
Remark: Two and only two tangents can be drawn to a circle from an external point.

Construction 4

To draw a circle circumscribing a given triangle.

Let *ABC* be a triangle. It is required to draw a circle circumscribing it. That is, a circle which passes through the three vertices *A, B* and *C* of the triangle *ABC* is to be drawn.

Construction :

(1) *EM* and *FN* the perpendicular bisectors of AB and *AC* respectively are drawn. Let the line segments intersect each other at *O*.

(2) A, O are joined. With O as centre and radius equal to OA, a circle is drawn. Then the circle will pass through the points A, B and C and this circle is the required circumcircle of ΔABC.

Proof : B, O and C, O are joined. The point O stands on EM, the perpendicular bisector of AB.

$\therefore$ $OA = OB$. Similarly, $OA = OC$

$\therefore$ $OA = OB = OC$.

Hence, the circle drawn with O as the centre and OA as the radius passes through the three points A, B and C. This circle is the required circumcircle of ΔABC.

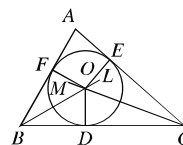
Activity:
In the above figure, the circumcircle of an acute angled triangle is constructed. Construct the circumcircle of an obtuse and right-angled triangles.

Notice that for in obtuse-angled triangle, the circumcentre lies outside the triangle, in acute-angle triangle, the circumcentre lies within the triangle and in right-angled triangle, the circumcentre lies on the hypotenuse of the triangle.

Construction 5

To draw a circle inscribed in a triangle.

Let ΔABC be a triangle. To inscribe a circle in it or to draw a circle in it such that it touches each of the three sides BC, CA and AB of the triangle.

Construction : BL and CM, the bisectors respectively of the angles $\angle ABC$ and $\angle ACB$ are drawn. Let the line segments intersect at O. OD is drawn perpendicular to BC from O and let it intersect BC at D. With O as centre and OD as radius, a circle is drawn. Then, this circle is the required inscribed circle.

Proof : From O, OE and OF are drawn perpendiculars respectively to AC and AB. Let these two perpendiculars intersect the respective sides at E and F. The point O lies on the bisector of $\angle ABC$.

$\therefore$ $OF = OD$.

Similarly, as O lies on bisector of $\angle ACB$, $OE = OD$

$\therefore$ $OD = OE = OF$

Hence, the circle drawn with centre as O and OD as radius passes through D, E and F.

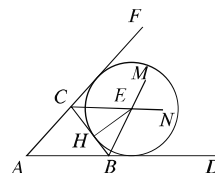
Again, BC, AC and AB respectively are perpendiculars to OD, OE and OF at their extremities. Hence, the circle lying inside ΔABC touches its sides at the points D, E and F.

Hence, the circle DEF is the required inscribed circle of ΔABC.

Construction 6

To draw an ex-circle of a given triangle.

Let ABC be a triangle. It is required to draw its excircle. That is, to draw a circle which touches one side of ABC and the other two sides produced.

Construction: Let AB and AC be produced to D and F respectively. BM and CN, the bisectors of $\angle DBC$ and $\angle FCB$ respectively are drawn. Let E be their point of intersection. From E, perpendicular EH is drawn on BC and let EH intersect BC at H. With E as centre and radius equal to EH, a circle is drawn.

The circle HGL is the excircle of the triangle ABC.

Proof : From E, perpendiculars EG and EL respectively are drawn to line segments BD and CF. Let the perpendicular intersect line segments G and L respectively. Since E lies on the bisector of $\angle DBC$

$\therefore EH = EG$

Similarly, the point E lies on the bisector of $\angle FCB$, so $EH = EL$

$\therefore EH = EG = EL$

Hence, the circle drawn with E as centre and radius equal to EL passes through H, G and L.

Again, the line segments BC, BD and CF respectively are perpendiculars at the extremities of EH, EG and EL. Hence, the circle touches the three line segments at the three points H, G and L respectively. Therefore, the circle HGL is the excircle of ΔABC.

Remark : Three excircles can be drawn with any triangle.

Activity: Construct the two other excircles of a triangle.

Exercise 8·5

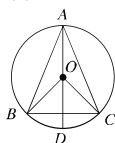
1. Observe the following information:

   i. The tangent to a circle is perpendicular to the radius to the point of contact.

   ii. The angle subtended in a semicircle is a right angle.

   iii. All equal chords of a circle are equidistant from the centre.

Which one of the following is correct ?

(a) i and ii (b) i and iii (c) ii and iii (d) i, ii and iii

Use the above figure to answer questions 2 and 3:

2. $\angle BOD$ equals to

   a. $\frac{1}{2}\angle BAC$ b. $\frac{1}{2}\angle BAD$ c. $2\angle BAC$ d. $2\angle BAD$

3. The circle is of the triangle ABC

   a. inscribed circle b. circumscribed circle c. excircle d. ellipse

4. The angle inscribed in a major arc is

   a. acute angle b. right angle c. obtuse angle d. complementary angle

5. Draw a tangent to a circle which is parallel to a given straight line.

6. Draw a tangent to a circle which is perpendicular to a given straight line.

7. Draw two tangents to a circle such that the angle between them is $60°$.

8. Draw the circumcircle of the triangle whose sides are 3 cm, 4 cm and 4.5 cm and find the radius of this circle.

9. Draw an excircle to an equilateral triangle ABC touching the side AC of the triangle, the length of each side being 5 cm.

10. Draw the inscribed and the circumscribed circles of a square.

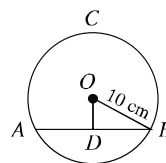
11. Prove that two circles drawn on equal sides of an isosceles triangle as diameters mutually intersect at mid point of its base.

12. Prove that in a right-angled triangle, the length of line segment joining mid point of the hypotenuse to opposite vertex is half the hypotenuse.

13. ABC is a triangle. If the circle drawn with AB as diameter intersects BC at D, prove that the circle drawn with AC as diameter also passes through D.

5, If the chords AB and CD of a circles with centre O intersect at an internal point E, prove that $\angle AEC = \frac{1}{2}(\angle BOD + \angle AOC)$.

6. AB is the common chord of two circles of equal radius. If a line segment meet through the circles at P and Q, prove that ΔOAQ is an isosceles triangle.

7. If the chord $AB = x$ cm and $OD \perp AB$, are in the circle ABC with centre O use the adjoint figure to answer the following questions:

a. Find the area of the circle.

b. Show that D is the mid point of AB.

c. If $OD = \left(\frac{x}{2} - 2\right)$ cm, determine x.

8. The lengths of three sides of a triangle are 4 cm, 5 cm and 6 cm respectively. Use this information to answer the following questions:

a. Construct the triangle.

b. Draw the circumcircle of the triangle.

c. From an exterior point of the circumcircle,, draw two tangents to it and show that their lengths are equal.